Nonspreading wave packets

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We show that for a wave ψ in the form of an Airy function the probability density $|\psi|^2$ propagates in free space without distortion and with constant acceleration. This "Airy packet" corresponds classically to a family of orbits represented by a parabola in phase space; under the classical motion this parabola translates rigidly, and the fact that no other curve has this property shows that the Airy packet is unique in propagating without change of form. The acceleration of the packet (which does not violate Ehrenfest's theorem) is related to the curvature of the caustic (envelope) of the family of world lines in spacetime. When a spatially uniform force $F(t)$ acts the Airy packet continues to preserve its integrity. We exhibit the solution of Schrödinger's equation for general $F(t)$ and discuss the motion for some special forms of $F(t)$.

I. INTRODUCTION

Dispersion in the Schrödinger equation (embodying the ability of classical particles to move at different speeds) suggests that all wave packets must change their form as they propagate[1] in free space. And Ehrenfest's theorem[1] (embodying Newton's second law for classical particles) suggests that no wave packet can accelerate in free space. It therefore comes as a surprise to discover a wave packet $\psi(x,t)$ whose probability density $|\psi(x,t)|^2$ not only remains unchanged in form but also continually accelerates, even though no force acts.

At $t = 0$ this wave packet is

$$\psi(x,0) = \mathrm{Ai}(Bx/\hbar^{2/3}), \tag{1}$$

where B is an arbitrary constant (taken as positive for convenience) and Ai denotes the Airy function,[2] whose square is sketched in Fig. 1. The "Airy packet" evolves according to the Schrödinger equation for a particle with mass m, namely

$$-\frac{\hbar^2}{2m}\frac{\partial^2 x}{\partial x^2} = i\hbar\frac{\partial \psi}{\partial t}, \tag{2}$$

whose solution is

$$\psi(x,t) = \mathrm{Ai}\left[\frac{B}{\hbar^{2/3}}\left(x - \frac{B^3t^2}{4m^2}\right)\right] e^{(iB^3t/2m\hbar)[x-(B^3t^2/6m^2)]}. \tag{3}$$

This is easily verified by direct substitution and use of the Airy function's differential equation.[2] Alternatively, we could build the solution (3) out of plane waves, by using the integral representation of the Airy function,[2] as follows:

$$\psi(x,t) = \frac{\hbar^{2/3}}{2\pi B}\int_{-\infty}^{\infty} dk\, e^{i(kx-\hbar k^2t/2m+\hbar^2k^3/3B^3)}.$$

It is clear from (3) that $|\psi|^2$ does indeed propagate without spreading, and accelerates to the right with velocity $B^3t^2/2m^2$.

In Sec. II we explain these two strange properties of the Airy packet. Both have a classical origin, and illustrate the fact that quantum wave functions correspond not to individual classical particles but to *families* of particle orbits. We shall show that what accelerates in the Airy packet is not any individual particle but the *caustic* (i.e., the envelope, or focus) of the family of orbits. Classical analysis of the trajectories will reveal that the nonspreading property of the Airy packet is unique (apart from the trivial plane wave, for which $|\psi|^2$ is independent of x).

In Sec. III we show that Airy packets continue to propagate without spreading when a spatially uniform force $F(t)$ acts, even if $F(t)$ has arbitrary time dependence. A constant $F(t)$ can reduce the wave to rest, and an oscillatory $F(t)$ stimulates the packet into a secular drift with superimposed oscillations.

II. CLASSICAL MECHANICS OF THE AIRY PACKET

At any instant t the family of orbits that is the classical counterpart of the Airy packet is represented by a curve $p = P_t(x)$ in the classical phase space whose variables are the coordinate x and momentum p. The evolution of the packet is mirrored classically by the way the curve changes as each point on it moves in accordance with Hamilton's equations.

To find the curve $P_0(x)$ corresponding to the initial packet given by Eq. (1), we employ the semiclassical approximation to the Airy function, obtained by considering $\hbar^{2/3}$ to be small (in comparison with $|Bx|$). Standard as-

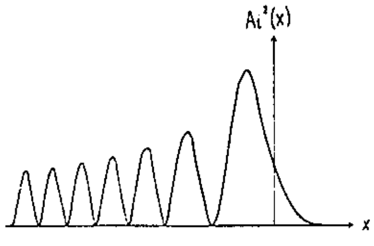

Fig. 1. Probability density for the Airy wave packet (1) with $B/\hbar^{2/3} = 1$.

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ymptotic forms[2] show that $\psi(x,0)$ is exponentially small for $x > 0$ and hence negligible, and oscillatory for $x < 0$, the precise expression being

$$\psi(x,0) \approx \frac{1}{\sqrt{\pi}}\left(\frac{\hbar^{2/3}}{-Bx}\right)^{1/4} \sin[\pi/4 + 2(-Bx)^{3/2}/3\hbar] \quad (4)$$

$(-Bx \gg \hbar^{2/3})$. This has a standard semiclassical form[3]

$$\psi(x,0) = \text{const}\left[\left(\frac{\partial^2 S_+}{\partial x^2}\right)^{1/2} e^{iS_+(x)/\hbar} + \left(\frac{\partial^2 S_-}{\partial x^2}\right)^{1/2} e^{iS_-(x)/\hbar}\right], \quad (5)$$

in which the actions $S_\pm$ are

$$S_\pm(x) = \pm(2/3)(-Bx)^{3/2}, \quad (6)$$

and according to a well-known prescription[3] the classical momenta corresponding to the point x are

$$p = P_0(x) = \frac{\partial S_\pm(x)}{\partial x} = \pm(-B^3x)^{1/2}. \quad (7)$$

Therefore at $t = 0$ the Airy packet corresponds to a family of classical orbits filling a parabola in phase space (Fig. 2). It will be more convenient to write this in the form

$$x = X_0(p) = -p^2/B^3. \quad (8)$$

To explain why the Airy packet is the only one that does not spread, we first note that a necessary condition for *any* probability density to propagate unchanged in form is that its semiclassical representation translates rigidly along x as time elapses. This in turn requires that the curve $P_t(x)$ or $X_t(p)$ representing the corresponding family of orbits translates rigidly in phase space as time elapses. (If this requirement were violated, that is if the curve were to rotate or deform, the x dependence of the multipliers $\partial^2 S/\partial x^2 = \partial P/\partial x$ in (5) would alter, thus distorting the semiclassical wave packet.) An initial point x_0,p_0 moves according to Hamilton's equations by

$$x = x_0 + p_0 t/m, \qquad p = p_0, \quad (9)$$

and this corresponds to *simple shear* of the phase space, with points on the x axis remaining fixed. Under this deformation the curve $X_0(p)$ changes to

$$x = X_t(p) = X_0(p) + pt/m. \quad (10)$$

Now, only two curves translate rigidly under this deformation. One is any straight line p = const, which corresponds to the trivial case of the plane wave $\psi = \exp(ipx/\hbar)$. The other is *any parabola whose symmetry axis is parallel to the x axis, which according to (8) corresponds to the*

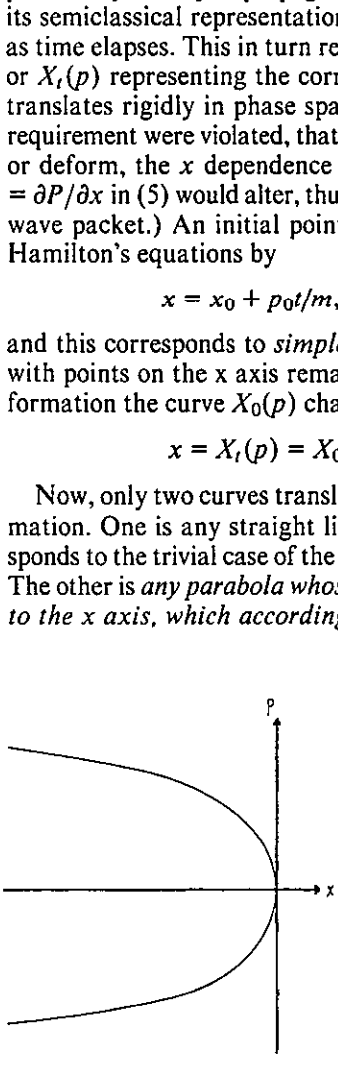

Fig. 2. Family of orbits at $t = 0$ fills a parabola in the classical phase space.

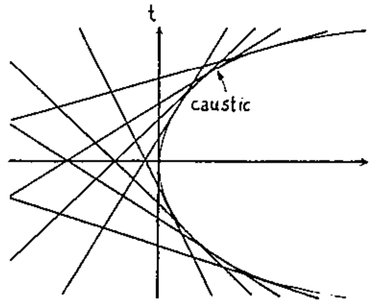

Fig. 3. Parabolic caustic enveloping straight world lines in force-free space time.

Airy packet. Explicitly, the shifted parabola at time t has the equation

$$x - B^3t^2/4m^2 = -(p - tB^3/2m)^2/B^3. \quad (11)$$

The proof that no other curve translates rigidly can be easily constructed by the interested reader by seeking curves which transform into themselves under shifts $a(t)$ and $b(t)$ in x and p, i.e., for which the substitutions

$$x = \xi - a(t), \qquad p = \pi - b(t), \quad (12)$$

transform (10) into

$$\xi = X_0(\pi). \quad (13)$$

This completes the explanation of why the Airy packet remains undistorted as it progresses.

Now we must understand how the Airy packet can accelerate even though no force acts on the particles. First of all, we remark that such acceleration does not contradict Ehrenfest's theorem,[1] according to which the center of gravity of a packet in free space moves with constant speed. The reason is that for the Airy packet the center of gravity cannot be defined, because the Airy function is not square integrable: it cannot represent the probability density for a single particle, but corresponds to an infinite number of particles, just like the plane wave and other wave functions in scattering theory. In fact the greatest value of $|\psi|^2$ lies close to the place (Fig. 1) where the argument of the Airy function (3) is zero, and according to equation (11) this corresponds to the x value for which the moving parabola has infinite slope, that is to the boundary of the classically allowed region. *This* is what accelerates.

On a spacetime diagram (Fig. 3) it is very clear how the classical boundary is the *caustic* (envelope) of the family of trajectories. For in x, t space the world lines of the trajectories (10) [with $X_0(p)$ given by (8)] are straight lines parameterised by p, and their envelope [obtained by eliminating p from (10) by differentiation] is just the parabola $x = t^2B^3/4m^2$. The acceleration of the classical boundary is embodied in the curvature of the caustic, and it is perfectly obvious that a family of straight world lines can be enveloped by a curved caustic. Further clarification of the apparent conflict with Ehrenfest's theorem is given in the appendix.

III. MOTION OF AN AIRY PACKET IN A TIME-VARYING SPATIALLY UNIFORM FORCE

Now let the initial wave (1) evolve not in free space but in a potential

$$V(x,t) = -F(t)x, \quad (14)$$

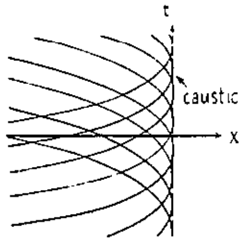

Fig. 4. Straight caustic enveloping parabolic world lines in the presence of a constant force.

representing a force $F(t)$. The solution of the corresponding time-dependent Schrödinger equation is

$$\psi(x,t) = \mathrm{Ai}\left[\frac{B}{\hbar^{2/3}}\left(x - \frac{B^3t^2}{4m^2} - \int_0^t \frac{d\tau(t-\tau)F(\tau)}{m}\right)\right] e^{i\phi(x,t)}, \quad (15)$$

where

$$\phi(x,t) = \frac{B^3t}{2m\hbar}\left(x - \frac{B^3t^2}{6m^2}\right) + \frac{x}{\hbar}\int_0^t d\tau F(\tau) - \frac{B^3}{2m^2\hbar} \times \int_0^t d\tau \int_0^\tau d\tau'(2\tau - \tau')F(\tau') - \frac{1}{2m\hbar}\int_0^t d\tau \left[\int_0^\tau d\tau' F(\tau')\right]^2. \quad (16)$$

The correctness of this solution can be verified by substitution. We actually found it in two different ways: (a) by evaluating Feynman's path integral[4] for the propagator of the Schrödinger equation, and then integrating over the initial wave (1); and (b) solving the Schrödinger equation in momentum representation using the method of characteristics.

The result (15) shows that once again the probability density $|\psi|^2$ of the Airy packet propagates without change of form. Its center moves along the trajectory given by

$$x_0(t) = \frac{B^3t^2}{4m^2} + \frac{1}{m}\int_0^t d\tau F(\tau)(t-\tau). \quad (17)$$

We now examine the motion resulting from some special forms of $F(t)$.

(i) The constant force

$$F(t) = -B^3/2m \quad (18)$$

gives $x_0(t) = 0$, i.e., such a force is just sufficient to overcome the "natural" tendency of the packet to accelerate. This result reproduces the well-known fact that the Airy function is a solution of the time-independent Schrödinger equation in a linear potential. In geometric language, the force causes the world lines on Fig. 3 to curve into parabolas, which can now envelope a straight caustic parallel to the t axis (Fig. 4), so that the classical boundary remains at rest.

(ii) If an impulse is added to the constant force (18), i.e., if

$$F(t) = -B^3/2m + mu\delta(t), \quad (19)$$

then

$$x_0(t) = ut, \quad (20)$$

i.e., the packet moves with constant speed u.

(iii) If instead of an impulse a sinusoidally oscillating force is added to (18), i.e., if

$$F(t) = -B^3/2m + F_0\cos(wt + \alpha), \quad (21)$$

then

$$x_0(t) = \frac{F_0}{m}\left[\frac{\cos wt - \cos(wt+\alpha)}{w^2} - \frac{t\sin\alpha}{w}\right]. \quad (22)$$

This causes the packet to oscillate, as expected, but the oscillations are superimposed on a secular drift with velocity $-F_0\sin\alpha/wm$. If $\alpha = 0$ the packet oscillates not about $x_0 = 0$ but about the point $x_0 = F_0\cos\alpha/mw^2$.

IV. DISCUSSION

We think the Airy packet is worth introducing into elementary quantum mechanics courses. Its unfamiliar properties, apparently contradicting the subject's folklore, provide a nontrivial illustration of the fact that a wave function corresponds to a family of orbits and not to a single particle. The unique nonspreading property is easily related to the unique shape of curve which is unaltered (apart from translation) as the classical motion shears the phase space. Moreover, the role played by the caustic shows dramatically how features of wave functions can be dominated by *forms* (envelopes of families of orbits, which can accelerate, even in empty space) rather than *things* (individual particles, which are constrained to move with constant velocity).

ACKNOWLEDGMENTS

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APPENDIX

Any square-integrable function constructed from Airy packets must obey Ehrenfest's theorem. One such function is the "eigendifferential"[5]

$$\chi(x,t) = \frac{B}{\hbar^{2/3}(\pi\sigma^2)^{1/4}}\int_{-\infty}^{\infty} dx'\, e^{-(x-x')^2/2\sigma^2}\,\psi(x',t), \quad (23)$$

where ψ is given by (3). $\chi(x,t)$ is normalized to unity. If σ is small χ is a superposition of Airy packets with a slight spread in the orgin. The resulting spread in the oscillations cancels by interference the oscillatory tail in $\psi(x,t)$ for infinitely negative x. By writing (23) as a Fourier integral it is not hard to show that the centre of gravity of the eigendifferential is at

$$\langle x\rangle \equiv \int_{-\infty}^{\infty} dxx'|\chi(x,t)|^2 = \frac{-\hbar}{2B^3\sigma^2}. \quad (24)$$

This shows that χ represents a physical situation in which the mean location is independent of time, in accord with Ehrenfest's theorem for this case where no forces act. The width of the eigendifferential is given by

$$\langle(x - \langle x\rangle)^2\rangle = \frac{\sigma^2}{2} + \frac{\hbar^4}{2B^6\sigma^4}\left(1 + \frac{t^2B^6\sigma^2}{m^2\hbar^2}\right). \quad (25)$$

For small σ this is constant for long times, but eventually the spreading due to the Gaussian cutoff takes over.

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